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The Shooting method for Solving Fuzzy Linear boundary Value Problems

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Abstract — In this paper, we apply the shooting method for solving the fuzzy boundary value problems of ordinary differential equations in connection with α – level sets(upper and lower α- levels)..

Keywords — Fuzzy Boundary Value Differential Equations, Shooting method.

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1 INTRODUCTION

CONSIDER the fuzzy boundary value problem of ordinary fuzzy differential equations which is given in the form:

$y'' = f(t, y, y'), a \leq t \leq$ (1)

together with fuzzy one of the following boundary conditions

1- $y(a) \simeq \tilde{\alpha}$, $y(b) \simeq$
2- $y'(a) \simeq \tilde{\alpha}$, $y'(b) \simeq$
3- $y(a) \simeq \tilde{\alpha}$, $y'(b) \simeq$
4- $a_0 y(a) + y'$ a) $\simeq$
5- $b_0 y(b) + b_1 y'(b) \simeq$

Where $a_0, a_1, b_0,$ are given constants and are fuzzy numbers

The linear second order fuzzy boundary value problem is:

$$y''(t) = p(t)y'(t) + q(t)y(t) + r$$

With boundary conditions given by either 1 or 2 or 3 or 4 above. For simplicity and illustration propose, we shall consider linear or nonlinear second order boundary value problems with fuzzy boundary conditions of type (1) only.

Here we will discuss the solution of such fuzzy boundary value problem with some illustrative examples and a comparison is given between the fuzzy solution when its=1 and the crisp or exact solution.[3] ,[4].

1.1 Fuzzy Number:

An arbitrary fuzzy number is represented by an order pair of functions $(\underline{y}(\alpha), \overline{y}(\alpha))$, for all $\alpha \in [0, 1]$; which satisfy the following requirements [6]; $\underline{y}(\alpha)$ and $\overline{y}(\alpha)$ are bounded, where $\underline{y}(\alpha) \leq \overline{y}(\alpha)$, $0 \leq \alpha \leq 1$, $\underline{y}(\alpha)$ and $\overline{y}(\alpha)$ refers to the left and right bounds of the fuzzy number.

Special cases:

Case (1),[5]:

Let $\tilde{y}(\alpha) = (\underline{y}(\alpha), \overline{y}(\alpha))$, $\alpha \in [0, 1]$ be a fuzzy number, we take:

$$yc(\alpha) = \frac{\overline{y}(\alpha) + \underline{y}(\alpha)}{2}, \quad yd(\alpha) = \frac{\overline{y}(\alpha) - \underline{y}(\alpha)}{2}$$

it is clear that $yd(\alpha) \geq 0$ and $\underline{y}(\alpha) = yc(\alpha) - yd(\alpha)$ and $\overline{y}(\alpha) = yc(\alpha) + yd(\alpha)$ also a fuzzy number $\tilde{y} \in E$ is said to be symmetric if $\tilde{y}(\alpha)$ is independent of α, for all $\alpha \in [0, 1]$.

Case (2), [5]:

Let $\tilde{y}(\alpha) = (\underline{y}(\alpha), \overline{y}(\alpha))$, $\tilde{v}(\alpha) = (\underline{v}(\alpha), \overline{v}(\alpha))$, $\alpha \in [0, 1]$ and also k, s are arbitrary real numbers. If $\tilde{w} = k\tilde{y} + s\tilde{v}$, then:

$wc(\alpha) = kyc(\alpha) + svc(\alpha)$
$wd(\alpha) = |k|yd(\alpha) + |s|vd(\alpha)$

Definition (1), [6]:

Let $\tilde{y}(\alpha) = (\underline{y}(\alpha), \overline{y}(\alpha))$, $\tilde{v}(\alpha) = (\underline{v}(\alpha), \overline{v}(\alpha))$, $\alpha \in [0, 1]$ be fuzzy numbers then the Hausdorff distance between $\tilde{y}$, $\tilde{v}$ is given by:

$$D(\tilde{y}, \tilde{v}) = \sup_{\alpha \in [0,1]} \max\{|\underline{y}(\alpha) - \underline{v}(\alpha)|, |\overline{y}(\alpha) - \overline{v}(\alpha)|\}$$

Case (3), [5]:

Clearly from case (2), we have:

$|\overline{y}(\alpha) - \overline{v}(\alpha)| \leq |yc(\alpha) - vc(\alpha)| + |yd(\alpha) - vd(\alpha)|$
$|\underline{y}(\alpha) - \underline{v}(\alpha)| \leq |yc(\alpha) - vc(\alpha)| + |yd(\alpha) - vd(\alpha)|$

Hence, for all $\alpha \in [0, 1]$:

$\max\{|\underline{y}(\alpha) - \underline{v}(\alpha)|, |\overline{y}(\alpha) - \overline{v}(\alpha)|\} \leq |yc(\alpha) - vc(\alpha)| + |yd(\alpha) - vd(\alpha)|$

and then:

$$D(\tilde{y}, \tilde{v}) = \sup_{\alpha \in [0,1]} \{|yc(\alpha) - vc(\alpha)| + |yd(\alpha) - vd(\alpha)|$$

Therefore, if $|yc(\alpha) - vc(\alpha)|$ and $|yd(\alpha) - vd(\alpha)|$ tends to zero hence $D(\tilde{y}, \tilde{v})$ tends to zero.

Let E be the set of all upper semi-continuous normal convex fuzzy numbers with bounded α-level intervals. This means that if $\tilde{v} \in E$, then the α-level set:

$[\tilde{v}]\alpha = \{s: \tilde{v}(s) \geq \alpha\}$

is closed bounded interval which is described by:

$[\tilde{v}]\alpha = [\underline{v}(\alpha), \overline{v}(\alpha)]$, for $\alpha \in [0, 1]$

and

$[\tilde{v}]0 = \overline{\bigcup_{\alpha \in [0,1]} [\tilde{v}]_\alpha}$

Two fuzzy numbers $\tilde{y}$ and $\tilde{v}$ are called equal, $\tilde{y} = \tilde{v}$, if $\tilde{y}(s) = \tilde{v}(s)$, for all $s \in R$ or $[\tilde{y}]\alpha = [\tilde{v}]\alpha$, for all $\alpha \in [0, 1]$.

Lemma (1), [7]:

If $\tilde{y}, \tilde{v} \in E$, then for $\alpha \in (0, 1]$:

$[\tilde{y} + \tilde{v}]\alpha = [\underline{y}(\alpha) + \underline{v}(\alpha), \overline{y}(\alpha) + \overline{v}(\alpha)]$

$[\tilde{y} . \tilde{v}]\alpha = [\min k, \max k]$

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Where $k = \{\underline{v}(\alpha)\,\underline{v}(\alpha),\ \underline{v}(\alpha)\,\overline{v}(\alpha),\ \overline{v}(\alpha)\,\underline{v}(\alpha),\ \overline{v}(\alpha)\,\overline{v}(\alpha)\}$.

Lemma (2), [7]:

Let $\tilde{v}(\alpha) = [\underline{v}(\alpha), \overline{v}(\alpha)]$, $\alpha \in (0, 1]$ be a given family of nonempty intervals. If:

$[\underline{v}(\alpha), \overline{v}(\alpha)] \supset [\underline{v}(s), \overline{v}(s)]$, for $0 < \alpha \leq s$.

$[\lim_{k\to\infty} \underline{v}(\alpha,k), \lim_{k\to\infty} \overline{v}(\alpha,k)] = [\underline{v}(\alpha), \overline{v}(\alpha)]$.

Whenever (α, k) is a non-decreasing sequence converging to $\alpha \in (0, 1]$, then the family $[\underline{v}(\alpha), \overline{v}(\alpha)]$, $\alpha \in (0, 1]$, represent the α-level sets of a fuzzy number v in E. Conversely, if $[\underline{v}(\alpha), \overline{v}(\alpha)]$, $\alpha \in (0, 1]$ are the α-level sets of a fuzzy number $\tilde{v} \in E$, then the conditions (i) and (ii) hold true.

Definition (2), [8]:

Let I be a real interval. A mapping $\tilde{v}: I \longrightarrow E$ is called a fuzzy process and we denote the α-level set by $[\tilde{v}(t)]\alpha = [\underline{v}(t,\alpha), \overline{v}(t,\alpha)]$.

The Seikkala derivative $\tilde{v}'(t)$ of $\tilde{v}$ is defined by:

$[\tilde{v}'(t)]\alpha = [\underline{v}'(t,\alpha), \overline{v}'(t,\alpha)]$

provided that this equation defines a fuzzy number $\tilde{v}(t) \in E$.

Definition (3), [8]:

The fuzzy integral of a fuzzy process $\tilde{v}$, denoted by $\int_a^b \tilde{v}(t)dt$, for $a, b \in I$, is defined by:

$$\left[\int_a^x \tilde{v}(t)\,dt\right]_\alpha = \left[\int_a^x \underline{v}(t,\alpha)\,dt, \int_a^x \overline{v}(t,\alpha)\,dt\right]$$

provided that the Lebsegue integrals exist

2 Solution of Fuzzy Cauchy Problems:

In this section we will discuss a method for solving homogenous linear systems of the first fuzzy differential equations.[2].

Consider the problem of solving the fuzzy linear homogenous differential system:

$$x' = Ax, x(0) \simeq \tilde{x}_0 \quad (2)$$

Where $x \in R^n$,A is $n \times n$ matrix and is the initial condition which is described by a vector made up of n-fuzzy numbers.

It is clear that a fuzzy number $\tilde{x}_0$, can be prescribed easily by its α-level sets, as:

$$[x_0]_\alpha = \{s: \mu_{\tilde{x}_0}(s) \geq \alpha\} \ ,0 \leq \alpha \leq 1$$

Due to the properties of the so defined fuzzy numbers, corresponds to an interval for each given value of α:

$$[x_0]_\alpha = [\underline{x}_{0_\alpha}, \overline{x}_{0_\alpha}]$$

Where $\underline{x}_{0_\alpha}$ and $\overline{x}_{0_\alpha}$ represents the lower and upper bounds of the fuzzy number,

[1].

Suppose that each element of the vector x in (2) at time t is a fuzzy number, where:

$$x_\alpha^k(t) = [\underline{x}_\alpha^k(t),\ \overline{x}_\alpha^k(t)]\ , k = 1,2 \ldots, n \quad (3)$$

It is shown that the evolution of the system(2) can be described by 2n-differential equations for the end points of the intervals, this is for each given time instant t and each value of $\alpha \in [0,1]$.Those equations for the end point of the intervals are:

$$\underline{x}'^k_\alpha(t) = \mathrm{Min}\left\{\sum_{j=1}^n a_{kj} u^j : u^j \left[\underline{x}_\alpha^j(t), \overline{x}_\alpha^j(t)\right]\right\}$$
$$\underline{x}'^k_\alpha(t) = \mathrm{Max}\left\{\sum_{j=1}^n a_{kj} u^j : u^j \left[\underline{x}_\alpha^j(t), \overline{x}_\alpha^j(t)\right]\right\} \quad (4)$$

With initial condition $\underline{x}_\alpha^k(0) = \underline{x}_{\alpha_0}^k$ and $\overline{x}_\alpha^k(0) = \overline{x}_{\alpha_0}^k$

Also, System (2) is linear, then equation (4) may be written:

$$\underline{x}'^k_\alpha(t) = \sum_{j=1}^n a_{kj} \quad (5)$$

Where

$$u^j = \begin{cases} \underline{x}_\alpha^j(t)\ ,\text{if } a_{kj} \geq 0 \\ \overline{x}_\alpha^j(t)\ \ ,\text{if } a_{kj} < 0 \end{cases}$$

and

$$\overline{x}'^k_\alpha(t) = \sum_{j=1}^n a_{kj} u^j \quad (6)$$

Where

$$u^j = \begin{cases} \overline{x}_\alpha^j(t)\ ,if \ \ a_{kj} \geq 0 \\ \underline{x}_\alpha^j(t)\ ,if\ a_{kj} < 0 \end{cases}$$

This means, for example, for any $\alpha \in [0,1]$ and k=1,2,3 (i.e.,3×3 system),then six differential equations will be obtained where two of them are for each k and each α related to one of end points, in other words:

$$x_\alpha^k(t) = \left[\underline{x}_\alpha^k(t),\ \overline{x}_\alpha^k(t)\right] = \begin{bmatrix} [\underline{x}_\alpha^1(t),\ \overline{x}_\alpha^1(t)] \\ [\underline{x}_\alpha^2(t),\ \overline{x}_\alpha^2(t)] \\ [\underline{x}_\alpha^3(t),\ \overline{x}_\alpha^3(t)] \end{bmatrix}$$

The method for solving directly the linear fuzzy system is meaningless; therefore an introduction of the representation of the fuzzy system using complex numbers is necessary.

In order to solve the fuzzy system of differential equations

$$x' = Ax, x(0) \simeq \tilde{x}_0$$

the utility of complex numbers will be used.

Recall that, there are two equations of the type (5) and (6) which could easily be written out explicitly.

Now, define new complex variables as follows

$$z_\alpha^k = \underline{x}_\alpha^k(t) + i\overline{x}_\alpha^k(t) \quad (7)$$

And the two operation carried on the complex numbers are:

(a) Identity operation which I given by e, such that:

$$ez_\alpha^k = z_\alpha^k \quad (8)$$

(b) the operator g corresponding to a flip about the diagonal in the complex plane, i.e.

$$g(z_\alpha^k) = g\left(\underline{x}_\alpha^k(t) + i\overline{x}_\alpha^k(t)\right) = \overline{x}_\alpha^k(t) + i\underline{x}_\alpha^k(t) \quad (9)$$

Where $g^2 = e$ and $g^k = e$ if k is even $g^k = g$ if k is odd, and therefore:

$$(ug)z_\alpha^k = (gu)z_\alpha^k \quad \text{for } u \in R \quad (10)$$

Using (7),(8) and(9),yield:

$$z_\alpha^k = \underline{x}_\alpha^k(t) + i\overline{x}_\alpha^k(t)$$

And hence

$$z'^k_\alpha = \underline{x}'^k_\alpha(t) + i\overline{x}'^k_\alpha(t)$$

But $\underline{x}'^k_\alpha(t) = \sum_{j=1}^n a_{kj} u^j$ and $i\,\overline{x}'^k_\alpha(t) = i\sum_{j=1}^n a_{kj} u^j$.

then

$$\underline{x}'^k_\alpha(t) + i\,\underline{x}'^k_\alpha(t) = \sum_{j=1}^n a_{kj} u^j + i\sum_{j=1}^n a_{kj} u^j$$

Hence:

$$z'^k_\alpha = a_{kj}(u^j + iu^j) = \begin{cases} a_{kj}(\underline{x}_\alpha^k + i\overline{x}_\alpha^k), \text{if } a_{kj} \ \geq 0 \\ a_{kj}(\overline{x}_\alpha^k + i\underline{x}_\alpha^k), \text{if } a_{kj} \ < 0 \end{cases}$$

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$$= \begin{cases} a_{kj}\ z^k_\alpha, & \text{if } a_{kj} \geq 0 \\ a_{kj}\ (gz^k_\alpha), & \text{if } a_{kj} < 0 \end{cases}$$

Now, using equation (10), whenever:]

$$z'^k_\alpha = \begin{cases} a_{kj}\ z^k_\alpha, & \text{if } a_{kj} \geq 0 \\ ga_{kj}\ z^k_\alpha, & \text{if } a_{kj} < 0 \end{cases}$$

and in order to simplify the last formula, let:

$$b_{ij} = \begin{cases} ea_{ij}, & \text{if } a_{ij} \geq 0 \\ ga_{ij}, & \text{if } a_{ij} < 0 \end{cases} \quad (11)$$

Then:

$$z'^k_\alpha = \begin{cases} b_{ij} z^k_\alpha, & \text{if } a_{kj} \geq 0 \\ b_{ij}(gz^k_\alpha), & \text{if } a_{kj} < 0 \end{cases}$$

with initial condition $z_\alpha(0) = z_{\alpha 0}$.

Now, $x' = Ax$, which has the solution $x = ce^{At}$ and since $x(0) = x_0$ then:

$$x(t) = x_0 e^{At}$$

Similarly:

$$z_\alpha(t) = z_{\alpha 0} e^{Bt} \quad (12)$$

but since the problem is to evaluate the exponential of the matrix B, then certain elements are multiplied by the flip operators (e and g, where $b_{ij} = ea_{ij}$ if $a_{ij} \geq 0$ and $b_{ij} = ga_{ij}$ if $a_{ij} < 0$ This can be achieved for small values of t writing the matrix B as the sum of two matrices, one of which is multiplied by the operator e and the other by g, for example:

$$B = eC + gD$$

And for small t, we have:

$$\exp(tB)\, z_{\alpha 0} = \exp\left(t(eC + gD)\right) z_{\alpha 0} = \exp(teC)\exp(tgD)\, z_{\alpha 0}$$

Where O(t) is a function of t, such that O(t)/t→0.

The first part $\exp(teC)$is simply the standard matrix exponential, because e is the identity operator. For the second part $\exp(tgD)$, noting that $g^k = e$ if k is even and $g^k = g$ if it is odd and then proceed to calculate the formal power series of $\exp(tgD)$ as follows:

$$\begin{aligned} \mathrm{Exp}(tgD)\, z_0 &= \left(I + tgD + \frac{t^2}{2!}\ g^2 D^2 + \frac{t^3}{3!}\ g^3 D^3 + \cdots\right) z_0 \\ &= \left(I + \frac{t^2}{2!}\ g^2 D^2\right) z_0 + \left(tgD + +\frac{t^3}{3!}\ g^3 D^3\right) z_0 \\ &= \left(I + \frac{t^2}{2!}\ D^2\right) z_0 + \left(tD + +\frac{t^3}{3!}\ D^3\right) gz_0 \\ &= \cosh(tD)\, z_0 + \sinh(tD)\ gz_0 \end{aligned}$$

Hence:

$$z_{\alpha 0}(t) = \exp(teC)\left(\cosh(tD)\, z_{\alpha 0} + \sinh(tD)\ g z_{\alpha 0}\right)$$

Let $\varphi(t) = \exp(tC)\cosh(tD)$

and $\Psi(t) = \exp(tC)\,\mathrm{Sinh}(tD)$. then:

$$z^k_\alpha = \varphi_{kj}(t) z^j_{\alpha 0} + \Psi_{kj}(t) g\, z^j_{\alpha 0}$$

But $z^k_\alpha = \underline{x}^k_\alpha(t) + i\overline{x}^k_\alpha(t)$, one get:

$$\underline{x}^k_\alpha(t) + i\overline{x}^k_\alpha(t) = \varphi_{kj}(t) z^j_{\alpha 0} + \Psi_{kj}(t) g\, z^j_{\alpha 0}$$

$$= \varphi_{kj}(t)\left(\underline{x}^j_{\alpha 0}(t) + i\ \overline{x}^j_{\alpha 0}(t)\right) + \Psi_{kj}(t)\left(\overline{x}^j_{\alpha 0}(t) + i\, \underline{x}^j_{\alpha 0}(t)\right)$$

Therefore:

$$\left.\begin{aligned} \underline{x}^k_\alpha(t) &= \textstyle\sum_{j=1}^n \varphi_{+kj}(t)\underline{x}^j_{\alpha 0}(t) + \Psi_{kj}(t)\ \overline{x}^j_{\alpha 0}(t) \\ \overline{x}^k_\alpha(t) &= \textstyle\sum_{j=1}^n \varphi_{kj}(t)\overline{x}^j_{\alpha 0}(t) + \Psi_{kj}(t)\ \underline{x}^j_{\alpha 0}(t) \end{aligned}\right\} \quad (13)$$

And hence the fuzzy solution is given in terms of its α -level sets as:

$$[x(t)]_\alpha = [\underline{x}_\alpha(t), \overline{x}_\alpha(t)]$$

3 The Shooting Method for Solving Fuzzy Boundary Value Problems:

The shooting method for solving fuzzy linear equation is based on the replacement of the fuzzy boundary value problem by its two related fuzzy initial value problems, as it is usual case in solving non-fuzzy boundary value problems.

Now, consider the linear second order fuzzy boundary value problem:

$$y'' = p(x)y' + q(x)y + r(x), a \leq x \leq b \quad (14)$$

$$y(a) \simeq \tilde{\alpha},\ \ y(b) \simeq \tilde{\beta} \quad (15)$$

Satisfying:

Hence, the related two fuzzy initial value problems are given by:

$$u'' = p(x)u' + q(x)u, a \leq x \leq b,\ \ u(a) \simeq \tilde{0},\ u'(a) \simeq \tilde{1} \quad (16)$$

And

$$v'' = p(x)v' + q(x)v + r(x), a \leq x \leq b, v(a) \simeq .\tilde{\alpha},\ \ v'(a) \simeq \tilde{0} \quad (17)$$

To find the solution of the fuzzy initial value problems (16) and (17), respectively, then the α-level equations these fuzzy differential equations are:

$$[u'']_\alpha = p(x)\ [u']_\alpha + q(x)[u]_\alpha, [u(a)]_\alpha \simeq \tilde{0}_\alpha, .[u'(a)]_\alpha \simeq \tilde{1}_\alpha \quad (18)$$

and

$$[v'']_\alpha = p(x)\ [v']_\alpha + q(x)[v]_\alpha + r(x),\ [v(a)]_\alpha \simeq \tilde{\alpha}_\alpha, .[v'(a)]_\alpha \simeq \alpha \quad (19)$$

Hence, for solution in the range $a \leq x \leq b$, where a and b are finite, suppose that the differential equations satisfies the existence and uniqueness theorem. The solution of equations.(18) and (19) are denoted, respectively, by:

$$[u(x)]_\alpha = \left[\underline{u}(x;\alpha), \overline{u}\ (x;\alpha)\right], \alpha \in [0,1]$$

and

$$[v(x)]_\alpha = \left[\underline{v}(x;\alpha), \overline{v}\ (x;\alpha)\right], \alpha \in [0,1]$$

Where $\underline{u}, \overline{u}, \underline{v}$ and $\overline{v}$ are the lower and upper nonfuzzy solution related to the fuzzy initial value problems at certain level. Also, the initial conditions can be rewritten as:

$$[u(a)]_\alpha = [\tilde{0}]_\alpha = \left[\underline{\tilde{0}}(\alpha), \overline{\tilde{0}}(\alpha)\right],$$

$$.[u'(a)]_\alpha = [\tilde{1}]_\alpha = \left[\underline{\tilde{1}}(\alpha), \overline{\tilde{1}}(\alpha)\right] \quad (20)$$

and

$$[v(a)]_\alpha = [\tilde{0}]_\alpha = \left[\underline{\tilde{\alpha}}(\alpha), \overline{\tilde{\alpha}}(\alpha)\right],$$

$$.[v'(a)]_\alpha = [\tilde{0}]_\alpha = \left[\underline{\tilde{0}}(\alpha), \overline{\tilde{0}}(\alpha)\right] \quad (21)$$

and hence the final solution of the fuzzy BVP can be obtained using previous discussed methods which are given for lower and upper cases by:

$$\begin{aligned} \underline{y}(x) &= \underline{v}(x) + \lambda\underline{u}(x) \\ \overline{y}(x) &= \overline{v}(x) + \lambda\overline{u}(x) \end{aligned} \quad (22)$$

Where: $\underline{\lambda} = \frac{\tilde{\beta} - \underline{v}(b)}{\underline{u}(b)}, \underline{u}(b) \neq 0$ and $\overline{\lambda} = \frac{\tilde{\beta} - \overline{v}(b)}{\overline{u}(b)}, \overline{u}(b) \neq 0$

Equation (9) is derived for the lower case, by letting

$$\underline{y}(x) = c_1\ \underline{v}(x) + c_2\lambda\underline{u}(x) \quad (23)$$

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and it is easily found that $c_1 = 1$ and $c_2 = \frac{\tilde{B}-\underline{v}(b)}{\underline{u}(b)}$ by using the boundary conditions at a and b .also, we can check that $\underline{y}(x)$ is really the solution of the original fuzzy BVP, since:

$$\underline{y}'(x) = \underline{v}'(x) + \frac{\tilde{B}-\underline{v}(b)}{\underline{u}(b)}\underline{u}'(x) \quad \text{and}$$

$$\underline{y}''(x) = \underline{v}'(x) + \frac{\tilde{B}-\underline{v}(b)}{\underline{u}(b)}\underline{u}''(x)$$

So:

$$\underline{y}''(x) = p(x)\underline{v}'(x) + q(x)\underline{v}(x) + r(x) + \frac{\tilde{B}-\underline{V}(b)}{\underline{u}(b)}(p(x)\underline{u}'(x) + q(x)\underline{u}(x))$$

$$= p(x)\left\{\underline{v}'(x) + \frac{\tilde{B}-\underline{v}(b)}{\underline{u}(b)}\underline{u}'(x)\right\} + q(x)\left\{\underline{v}(x) + \frac{\tilde{B}-\underline{v}(b)}{\underline{u}(b)}\underline{u}(x)\right\} + r(x)$$

$$= p(x)\underline{y}' + q(x)\underline{y}(x) + r(x)$$

Moreover:

$$\underline{y}(a) = \underline{v}(a) + \frac{\tilde{B}-\underline{v}(b)}{\underline{u}(b)}\underline{u}(a) = \tilde{\alpha} + \frac{\tilde{B}-\underline{v}(b)}{\underline{u}(b)} \times 0 = \tilde{\alpha}$$

And

$$\underline{y}(b) = \underline{v}(b) + \frac{\tilde{B}-\underline{v}(b)}{\underline{u}(b)}\underline{u}(b) = \underline{v}(b) + \tilde{\beta} - \underline{v}(b) = \tilde{\beta}$$

Hence, $\underline{y}(x)$ is the unique solution to the linear BVP, provided of course, that $\underline{u}(b) \neq 0$

Similarly, as in upper case, we have:

$$\bar{y}(x) = \bar{v}(x) + \frac{\tilde{B}-\bar{v}(b)}{\bar{u}(b)}\bar{u}(x)$$

The following example illustrates the above discussion:

Example:

To solve the non- homogeneous fuzzy BVP:

$$y'' = \frac{-2}{x}y' + \frac{2}{x^2}y + x^2, 1 \le x \le 2$$

$$y(1) \simeq \tilde{1}, \quad y(2) \simeq \tilde{2}$$

By using the linear shooting method. This problem has the exact crisp solution:

$$y(x) = -\frac{1}{8}x + \frac{1}{x^2} + \frac{1}{8}x^4$$

now, to solve the homogeneous problem:

$$u'' = \frac{-2}{x}u' + \frac{2}{x^2}u, \; u(1) \simeq \tilde{0}, \quad u'(1) \simeq \tilde{1}$$

Let $u_1 = u$, then $u' = u_2$ and so $u'_2 = \frac{-2}{x}u_2 + \frac{2}{x^2}u_1$. therefore in matrix form: $\begin{bmatrix} u'_1 \\ u'_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{2}{x^2} & \frac{-2}{x} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

So, the desired system of homogeneous system of initial value problem, and carrying out similar whenever x=2, we have for all $\alpha \in [0,1]$:

$$\underline{u}_\alpha^{(1)}(2) = 2a + \frac{22}{3}c - \frac{20}{3}d, \quad = \frac{-20}{3}c + 2b + \frac{22}{3}d$$

and

$$\underline{u}_\alpha^{(2)}(2) = a + \frac{22}{3}c - \frac{20}{3}d, \quad \bar{u}_\alpha^{(2)}(2) = \frac{-20}{3}c + b$$

where $a = -\sqrt{1-\alpha}$,$b = \sqrt{1-\alpha}$,$c = 1-\sqrt{1-\alpha}$, . $d = 1+\sqrt{1-\alpha}$

to check the validity of the results, if $\alpha = 1$, then $a = 0, b = 0, c = 1, d = 1$,and hence:

$$\underline{u}_\alpha^{(1)}(2) = \bar{u}_\alpha^{(1)}(2) = 0.666666 \quad \text{and}$$

$$\underline{u}_\alpha^{(2)}(2) = \bar{u}_\alpha^{(2)}(2) = 0.666666$$

Also, for the non-homogeneous problem:

$$v'' = \frac{-2}{x}v' + \frac{2}{x^2}v + x^2, \; v(1) \simeq \tilde{1}, \quad v'(1) \simeq \tilde{0}$$

Which has an equivalent matrix form:

$$\begin{bmatrix} v' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{2}{x^2} & \frac{-2}{x} \end{bmatrix}\begin{bmatrix} v \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ x^2 \end{bmatrix} = \begin{bmatrix} F \\ G \end{bmatrix}$$

So, the desire non-homogeneous system of differential with fuzzy initial conditions could also be solved. Solving the non-homogeneous system using Euler's method given in parametric form for $v(x)$ and $y(x)$. In this case is given as follows:

$$\left.\begin{array}{l} \underline{v}'(x,\alpha) = \underline{y}(x,\alpha), \; \bar{v}'(x,\alpha) = \bar{y}(x,\alpha) \\ \underline{y}'(x,\alpha) = \frac{-2}{x^2}\underline{v} - \frac{2}{x}\underline{y} + x^2, \; \bar{y}'(x,\alpha) = \frac{-2}{x^2}\bar{v} - \frac{2}{x}\bar{y} + x^2 \end{array}\right\} \quad (24)$$

With initial conditions are given for all $\alpha \in [0,1]$, by:

$\underline{v}(1,\alpha) \simeq \tilde{1}, \quad \bar{v}(1,\alpha) \simeq \tilde{1}$

$\underline{y}(1,\alpha) \simeq \tilde{0}, \quad \bar{y}(1,\alpha) \simeq \tilde{0}$

Using first order explicit Euler method:

$$\left.\begin{array}{l} \underline{v}_{n+1}(\alpha) = \underline{v}_n(\alpha) + hF_1\left(x_n, \underline{v}_n(\alpha), \underline{y}_n(\alpha)\right) \\ \bar{v}_{n+1}(\alpha) = \bar{v}_n(\alpha) + hF_2\left(x_n, \bar{v}_n(\alpha), \bar{y}_n(\alpha)\right) \end{array}\right\} \quad (25)$$

And

$$\left.\begin{array}{l} \underline{y}_{n+1}(\alpha) = \underline{y}_n(\alpha) + hG_1\left(x_n, \underline{v}_n(\alpha), \underline{y}_n(\alpha)\right) \\ \bar{y}_{n+1}(\alpha) = \bar{y}_n(\alpha) + hG_2\left(x_n, \bar{v}_n(\alpha), \bar{y}_n(\alpha)\right) \end{array}\right\} \quad (26)$$

Using equation (11) with equations (12) and (13), we get:

$$\left.\begin{array}{l} \underline{v}_{n+1}(\alpha) = \underline{v}_n(\alpha) + h\underline{y}_n(\alpha) \\ \bar{v}_{n+1}(\alpha) = \bar{v}_n(\alpha) + h\bar{y}_n(\alpha) \end{array}\right\} \quad (27)$$

And

$$\left.\begin{array}{l} \underline{y}_{n+1}(\alpha) = \underline{y}_n(\alpha) + h\left(\frac{2}{x_n^2}\underline{v}_n - \frac{2}{x_n}\underline{y}_n + x_n^2\right) \\ \bar{y}_{n+1}(\alpha) = \bar{y}_n(\alpha) + h\left(\frac{2}{x_n^2}\bar{v}_n - \frac{2}{x_n}\bar{y}_n + x_n^2\right) \end{array}\right\} \quad (28)$$

Hence, by using the first problem of equations (14) and (15), we get the following results which in table (1), for all $x \in [1,2], h = 0.1$ and $\alpha = 1$.

TABLE 1

x_n	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
$\bar{v}_n$	1	1.03	1.08	1.15	1.24	1.34	1.46	1.60	1.76	1.95
$\bar{y}_n$	0.3	0.53	0.72	0.90	1.08	1.27	1.47	1.68	1.91	2.16

The lower solution of equations (14) and (15) and by using the second problem of equations (14) and (15) we get the following results which are given in table (2) for all $x \in [1,2], h = 0.1$ and $\alpha = 1$.

TABLE 2

x_n	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
$\bar{v}_n$	1	1.03	1.08	1.15	1.24	1.34	1.46	1.60	1.76	1.95
$\bar{y}_n$	0.3	0.53	0.72	0.90	1.08	1.27	1.47	1.68	1.91	2.16

The upper solution of equations (14) and (15), where:

$$\underline{\lambda} = \frac{\tilde{B}-\underline{v}(b)}{\underline{u}(b)} = \frac{2-1.95}{0.666666} = 0.075 \quad \text{and}$$

$$\bar{\lambda} = \frac{\tilde{B}-\bar{v}(b)}{\bar{u}(b)} = \frac{2-1.95}{0.666666} = 0.075$$

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As a results, the general solution of the fuzzy BVP using the linear shooting method is given by:

$$\underline{y}(x_i) = \underline{v}(x_i) + \underline{\lambda}\, \underline{u}(x_i)$$
$$\overline{\overline{y}}(x_i) = \overline{v}(x_i) + \overline{\lambda}\overline{u}(x_i)$$

The result may be checked by comparing the solution with the crisp solution at $\alpha = 1$ and for $x = b = 2$, we have:

$\underline{y}(2) = \underline{v}(2) + \underline{\lambda}\, \underline{u}(2)$=1.95+0.075×0.666666=2

$\overline{\overline{y}}(2) = \overline{v}(2) + \overline{\lambda}\overline{u}(2)$= 1.95+0.075×0.666666=2

Clearly $\underline{y}(x)$ and $\overline{y}(x)$ are equal only when $\alpha = 1$.

The results of the calculations using N=10 and h=0.1 are given in table (3)

TABLE 3

x_i	$\underline{y}(x_i)$	$\overline{y}(x_i)$	Exact solution
1	1	1	1.000
1.1	1.012	1.012	0.872
1.2	1.121	1.121	0.804
1.3	1.240	1.240	0.786
1.4	1.353	1.353	0.815
1.5	1.458	1.458	0.890
1.6	1.567	1.567	1.010
1.7	1.671	1.671	1.178
1.8	1.785	1.785	1.396
1.9	1.891	1.891	1.669
2	2	2	2.000

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